It is found that the effects of stream nonuniformity on the induced drag are significant in a wide range of the parameters. The variations of the drag factor k with the velocity ratio U_0/U_I or U_I/U_2 become stronger as the scale ratio s/h increases. For the jet and wake streams the effects are much larger than for the linearly sheared stream; this is due to the dependence of the solution for lift distribution on the second derivative of stream profile $U''(0)$, which was discussed in Ref. 1.

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Simple and Accurate Calculation of Supersonic Nozzle Contour

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Introduction

THE method of characteristics is commonly used to design a supersonic nozzle.[1] This method is widely applied to large nozzles where the boundary-layer displacement thickness is small compared to the nonviscous flow. As the nozzle becomes larger and the Mach number higher, one should use more characteristics lines in order to achieve an accurate supersonic profile. Such a profile is necessary to obtain a uniform flowfield at the exit of the nozzle in the supersonic test chamber. A simple method that considerably increases the accuracy of the solution for a given number of characteristics lines is described in this paper.

The Method

Let us assume a straight sonic line at the throat of a supersonic nozzle, perpendicular to the flow direction. To calculate the flow in the supersonic region, the characteristics lines are taken as straight segments between two grid points.[1] There are few procedures that take the curvature of the lines into consideration.[1-3] However, these procedures give systematical error in the calculated supersonic nozzle wall profile that is usually overcome by increasing the number of characteristics lines.[2] One way to check the accuracy of the calculation is to compare the final area ratio of the supersonic nozzle that is obtained by the method of characteristics to the one-dimensional area ratio for isentropic flow with the same specific heat and final Mach number. These two solutions should coincide since the flow is assumed to be uniform and the cross-sectional area perpendicular to the flow direction at both the throat and the nozzle exit. It is suggested that the same procedure be applied for each segment of the expansion waves prior to the construction of the grid in the characteristics calculation, as opposed to the procedure of determining the contour by streamlines for a given grid.[4] Following this method, one gets a supersonic nozzle wall boundary contour that is an envelope of the accurate one.

To explain the method, we consider the simple case of isentropic expansion flow near a convex corner, as shown in Fig. 1. Assuming that the flowfield outside the convex corner region is supersonic and uniform at Mach numbers M_1 and M_2 before and after the curve, respectively, the expansion waves are straight. Since there is no characteristics length to define a scale in the configuration perpendicular to the streamlines, the flow parameters must be constant along Mach line "rays" that are initiated at the corner. For each Mach number M, one defines a Mach line with an angle μ with respect to the flow direction at a particular point of the flow: $\mu = \sin^{-1}(1/M)$. Let μ_1 and μ_2 be the Mach angles corresponding to the Mach numbers M_1 and M_2 at the boundaries of the expansion fan, and θ the angle change in the flow direction. Using the method of waves it is desired to represent the expansion fan by a single Mach line so that the incoming and outgoing streamlines will be straight. However, none of the μ_1 and μ_2, or the averaging procedures used,[3] can keep the streamlines outside the expansion fan unchanged. The relation between the Prandtl-Meyer functions ν_1 and ν_2 that correspond to M_1 and M_2, respectively, and θ, in simple isentropic turns, implies that $\nu_2 - \nu_2 = \theta$ where

$$\nu(M) = [(\gamma+1)/(\gamma-1)]^{1/2}\tan^{-1}$$
$$\times [(\gamma-1)(M^2-1)/(\gamma+1)]^{1/2} - \tan^{-1}(M^2-1)^{1/2}$$

where θ is the curve angle, the angle between the direction of a streamline before and after the curve region, and γ the ratio of specific heats.

Since θ, γ, and M_1 are defined, one can calculate M_2. On the other hand, since the flow is isentropic and since y_1 and y_2 are measured perpendicular to the flow direction, one can use the continuity equation for the flow that is limited between two streamlines, as is done in the one-dimensional flow. Thus,

$$\frac{y_2}{y_1} = \frac{M_1}{M_2}\left[\left(1+\frac{\gamma-1}{2}M_2^2\right)\Big/\left(1+\frac{\gamma-1}{2}M_1^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Therefore, for a given M_1, M_2, γ, and θ, y_2 is defined for each y_1. The streamlines are curved inside the expansion fan and straight outside. The next step is to find the intersection of the two straight lines that corresponds to the same streamline, before and after the expansion fan, inside the fan region. These two lines inside the expansion fan will be called the streamline envelope.

Since the streamlines outside the expansion fan are straight and parallel, their intersections define a straight line. An angle ω is defined between the new "Mach line" and the direction of the flow before it enters the expansion fan. ω can be found from the expression

$$y_1/y_2 = \sin(\omega)/\sin(\omega+\theta)$$

The new "Mach line" represents the flow inside the expansion fan, keeping the streamlines unchanged outside the fan region, and defines straight streamlines envelopes inside the fan. This procedure can also be applied to the wall nozzle boundary, since it also defines a streamline.

To calculate a contour of a supersonic nozzle wall boundary, a regular characteristics method should be followed.[3] In this procedure one divides the deflection angle into equal segments. However, instead of taking the inclination of each wave at angle μ, the new "Mach line" (at angle ω) should be taken. It keeps both the positions and slopes of the streamlines on the boundaries of each segment unchanged. Figure 2 describes the envelope of a supersonic nozzle wall boundary produced by one and two characteristics calculations for a sharp corner nozzle. The contours are so

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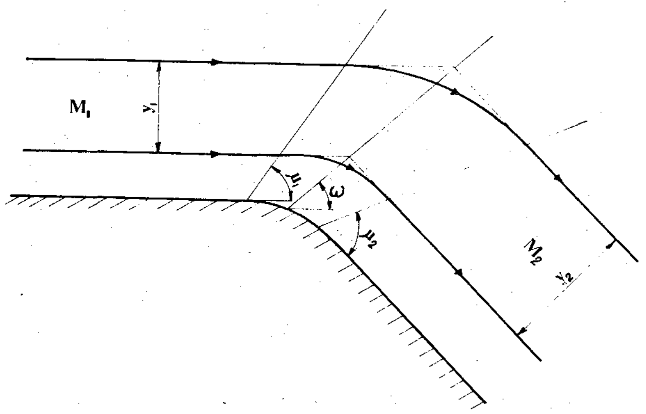

Fig. 1 Flow parameters of isentropic expansion flow near a curve convex corner.

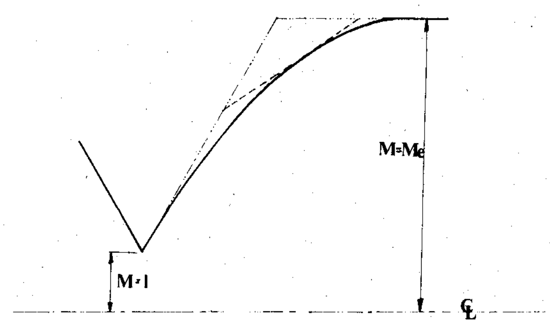

Fig. 2 Envelope of supersonic wall boundaries produced by one (– .. –) and two (– – –) characteristics for a sharp corner nozzle.

designed that each expansion wave should be canceled when it first hits the wall boundaries, as is done in the regular characteristics calculation. For the Laval nozzle, the segmented envelope sections are inside the nozzle wall boundaries of the supersonic flow up to the inflection point and are outside the nozzle contour after it. For the sharp corner nozzle, the segmented envelope sections are always on the outside of the supersonic nozzle wall boundaries.

Discussion of Results

To demonstrate the accuracy of this procedure, the wall contour produced by this method is compared to the wall contour produced by the regular method.[3,4] The Euler prediction corrector method is used as the regular method. The "true" wall profile is calculated by a large number of characteristics lines ($N=1000$). For the regular method the end sections of each nozzle wall segment are taken as the accurate value. For the present method the midpoint of each straight wall envelope section is taken as the accurate value. The middle points are close enough to the shortest distances between the "true" profile and the calculated segments. Since the last section of the envelope produced by the present method is always parallel to the flow direction at the centerline, the final area ratio always coincides with the one-dimensional calculation. The largest relative error in the present procedure is somewhere in the middle of the wall contour. For the regular method, the relative error of the area ratio at the nozzle exit is about the same as for the center of the wall contour.

To minimize the number of independent parameters, a sharp corner supersonic nozzle with a straight sonic line at the throat is chosen. One of the interior streamlines of the sharp corner nozzle may be employed for the wall contour to produce a Laval nozzle. There are two independent parameters that one has to select: γ and the final Mach

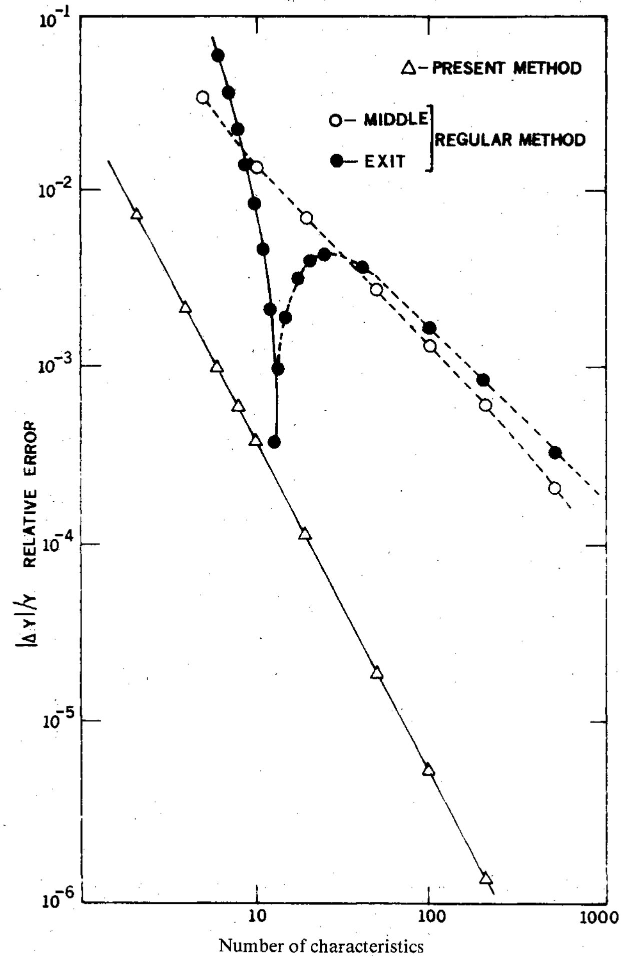

Fig. 3 Errors of the area ratios as a function of the number of characteristics lines that are used to calculate a sharp corner nozzle, ($\gamma=1.4$, $M_e=5$): Δ present method, ○ regular method for wall points of the center characteristics lines, ● final area ratio for the regular method (solid lines for positive errors, dashed lines for negative errors).

number M_e. The values chosen for this demonstration are: $\gamma=1.4$ and $M_e=5.0$. Comparison between the relative errors of the two methods is given in Fig. 3. The order R of the accuracy of the characteristics calculation is defined by

$$R=\log(|\Delta y|/y)/\log(\Delta N)$$

where $|\Delta y|/y$ is the relative error change in the area ratio and ΔN the change in the number of characteristics lines used to calculate the nozzle. For the regular method $R=1$, for the present method $R=2$. Moreover, for small number of characteristics lines the error for the present method is smaller. The accuracy of the present method for the same number of characteristics lines is always better than the regular method for each calculated point of the wall contour.

Other factors that should be taken into account to obtain a uniform flowfield at the nozzle end, such as boundary layers, are beyond the scope of this paper.

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